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DIFFERENTIAL GAMES WITH ELLIPSOIDAL PENALTIES†

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Antagonistic linear differential games in a fixed time interval are considered. The performance index of the games considered consists of the terminal term and integral penalties imposed on the players' controls. The terminal term is a quadratic form in the values of the phase vector at the final instant of time. The graphs of the penalty functions are halves of ellipsoid surfaces. It is proved that differential games of the class considered have a saddle point in the class of programmed strategies. Explicit expressions are obtained for optimal programmed strategies in terms of the vector of conjugate variables. Effective algorithms are presented for computing the vector of conjugate variables and it is proved that these algorithms converge. A regular approximately optimal strategy is constructed for differential games with purely geometrical constraints on the pursuer's control. As an example, a differential game in four-dimensional space is considered.

The analysis of differential games in which geometrical constraints are imposed on the players' controls is, technically speaking, an extremely complicated task. Algorithms for solving the problem in the general case are extremely ineffective [1, 2]. In this paper, we continue the study, begun in [3], of effective algorithms for constructing guaranteed strategies for the players on the assumption that the sets of admissible controls are ellipsoids. Unlike [3], where the strategies considered were guaranteed but not in general optimal, the algorithms considered here will construct optimal guaranteed strategies.

1. THE SADDLE-POINT THEOREM

Let us consider a differential game

$$\dot{x}(t) = A(t)x(t) + B_u(t)u(t) + B_\upsilon(t)\upsilon(t), \quad x(0) = x_0 \tag{1.1}$$

over the time interval $t \in [0; \vartheta]$ with performance index

$$J = \frac{1}{2}x^T(\vartheta)Fx(\vartheta) + \int_0^\vartheta (-\beta_u(t, u(t)) + \beta_\upsilon(t, \upsilon(t)))dt \tag{1.2}$$

$$\beta_u(t, u) = \gamma_u(t)\sqrt{1 - u^T G_u(t)u}, \quad \beta_\upsilon(t, \upsilon) = \gamma_\upsilon(t)\sqrt{1 - \upsilon^T G_\upsilon(t)\upsilon} \tag{1.3}$$

where $u(t) \in \mathbb{R}^p$ and $\upsilon(t) \in \mathbb{R}^q$ are the players' control and $x(t) \in \mathbb{R}^n$ is the phase vector of the system.

The following are assumed to be given:

(1) a symmetrical matrix $F \in \mathbb{R}^{n\times n}$;

(2) piecewise-continuous matrix-valued functions A: $[0; \vartheta] \to \mathbb{R}^{n\times n}$, B_u: $[0; \vartheta] \to \mathbb{R}^{n\times p}$, B_υ: $[0; \vartheta] \to \mathbb{R}^{n\times q}$, G_u: $[0; \vartheta] \to \mathbb{R}^{p\times p}$, G_υ: $[0; \vartheta] \to \mathbb{R}^{q\times q}$, such that for any $t \in [0, \vartheta]$ the matrices $G_u(t)$ and $G_\upsilon(t)$ are symmetric and positive-definite;

(3) piecewise-continuous functions γ_u: $[0; \vartheta] \to \mathbb{R}$, γ_υ: $[0; \vartheta] \to \mathbb{R}$ with positive values.

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The players' controls are subject to the geometrical constraints

$$u^T(t)G_u(t)u(t) \le 1, \quad \upsilon^T(t)G_\upsilon(t)\upsilon(t) \le 1, \quad t \in [0; \vartheta]$$

The players know the parameters of the game: ϑ, x_0, A, B_u, B_υ, F, G_u, G_υ, γ_u, γ_υ. At each instant of time t the players know the actual value of the phase vector $x(t)$. The aim of player u (player υ) is to minimize (maximize) the function J.

Since the graphs of the functions $u \mapsto \beta_u(t, u)$ and $\upsilon \mapsto \beta_\upsilon(t, \upsilon)$ lie on the surfaces of ellipsoids, the function β_u and β_υ will be called ellipsoidal penalties.

Let $\mathbb{R}^s$ denote the space of s-dimensional column vectors, and $\mathbb{R}^{m\times n}$ the space of $m \times n$ matrices. If $x \in \mathbb{R}^n$, we let $|x|$ denote the Euclidean norm of the vector x: $|x| = \sqrt{x^T x}$. The operator norm of a matrix $A \in \mathbb{R}^{m\times n}$ will be denoted by $||A||$: $||A|| = \max\limits_{x \in \mathbb{R}^n:\, |x| = 1} |Ax|$; L_s^2 is the space of square integrable functions (in the sense of Lebesgue integration) φ: $[0; \vartheta] \to \mathbb{R}^s$ and U is the set of admissible programmed strategies of player u:

$$U = \{u \in L_p^2 \colon u^T(t)G_u(t)u(t) \le 1 \text{ almost everywhere in } [0; \vartheta]\} \tag{1.4}$$

Similar, V will denote the set of set of admissible programmed strategies of player υ.

For any $u \in U$ and $\upsilon \in V$, we let $J(u, \upsilon)$ denote the value of the performance index (1.1)–(1.3) of the game corresponding to programmed controls u and υ.

The quadratic function $x^T F x$ can be reduced to canonical form. Matrices $S \in \mathbb{R}^{m\times n}$ and $F_1 \in \mathbb{R}^{m\times m}$ exist such that $F = S^T F_1 S$, where

$$F_1 = \begin{Vmatrix} E_r & 0 \\ 0 & -E_s \end{Vmatrix}, \quad r + s = m \tag{1.5}$$

Here m is the rank of the matrix F, and E_k is the $k \times k$ identity matrix.

Let the matrix-valued function Φ: $[0; \vartheta] \to \mathbb{R}^{m\times n}$ be a solution of the Cauchy problem

$$\dot{\Phi}(t) = -\Phi(t)A(t), \quad \Phi(\vartheta) = S \tag{1.6}$$

In particular, if $A(t) = A$ is a constant matrix, then $\Phi(t) = Se^{A(\vartheta - t)}$.

Replacing the phase vector $x(t)$ by the vector $z(t) = \Phi(t)x(t)$, w reduce the differential game (1.1)–(1.3) to a differential game with simple dynamics

$$\dot{z}(t) = \tilde{B}_u(t)u(t) + \tilde{B}_\upsilon(t)\upsilon(t), \quad z(0) = z_0 \tag{1.7}$$

where

$$\tilde{B}_u(t) = \Phi(t)B_u(t), \quad \tilde{B}_\upsilon(t) = \Phi(t)B_\upsilon(t), \quad z_0 = \Phi(0)x_0$$

and performance index

$$J = \frac{z^T(\vartheta)F_1 z(\vartheta)}{2} + \int_0^\vartheta (-\beta_u(t, u(t)) + \beta_\upsilon(t, \upsilon(t)))dt \tag{1.8}$$

We define the matrices

$$P_u(t) = \Phi(t)B_u(t)G_u^{-1}(t)B_u^T(t)\Phi^T(t), \quad \bar{P}_u = \int_0^\vartheta \frac{1}{\gamma_u(t)} P_u(t)dt \tag{1.9}$$

and similarly matrices $P_\upsilon(t)$, $\bar{P}_\upsilon$, For any $u \in U$, we define an m-dimensional column vector

$$\mathcal{B}(u) = \int_0^\vartheta \tilde{B}_u(t)u(t)dt$$

Lemma 1.1. Let the matrix $\bar{P}_u + \bar{P}_u F_1 \bar{P}_u$ be positive-semidefinite. Then for any function $u \in L_p^2$

$$\int_0^{\vartheta} \gamma_u(t) u^T(t) G_u(t) u(t) dt + \mathcal{B}^T(u) F_1 \mathcal{B}(u) \geq 0$$

Proof. Let the function $u \in L_p^2$ be given. Consider the problem of minimizing the functional

$$\bar{u} \mapsto \int_0^{\vartheta} \gamma_u(t) \bar{u}^T(t) G_u(t) \bar{u}(t) dt$$

over all $\bar{u} \in L_p^2$ such that $\mathcal{B}(\bar{u}) = \mathcal{B}(u)$. Since the matrix $\gamma_u(t) G_u(t)$ is positive-definite and the set $\{\bar{u} \in L_p^2: \mathcal{B}(\bar{u}) = \mathcal{B}(u)\}$ is non-empty, the problem has a solution u_0. Using Lagrange multipliers, we will show that a vector $\lambda \in \mathbb{R}^m$ exists such that the function u_0 is a stationary point of the functional

$$\bar{u} \mapsto \int_0^{\vartheta} \gamma_u(t) \bar{u}^T(t) G_u(t) \bar{u}(t) dt - 2\lambda^T(\mathcal{B}(\bar{u}) - \mathcal{B}(u))$$

that is

$$\gamma_u(t) G_u(t) u_0(t) - \tilde{B}_u^T(t) \lambda = 0$$

Consequently

$$u_0(t) = \frac{1}{\gamma_u(t)} G_u^{-1}(t) \tilde{B}_u^T(t) \lambda$$

Thus

$$\int_0^{\vartheta} \gamma_u(t) u^T(t) G_u(t) u(t) dt \geq \int_0^{\vartheta} \gamma_u(t) u_0^T(t) G_u(t) u_0(t) dt =$$

$$= \lambda^T \left(\int_0^{\vartheta} \frac{1}{\gamma_u(t)} \tilde{B}_u(t) G_u^{-1}(t) \tilde{B}_u^T(t) dt \right) \lambda = \lambda^T \bar{P}_u \lambda$$

where the vector λ satisfies the condition

$$\bar{P}_u \lambda = \left(\int_0^{\vartheta} \frac{1}{\gamma_u(t)} \tilde{B}_u(t) G_u^{-1}(t) \tilde{B}_u^T(t) dt \right) \lambda = \int_0^{\vartheta} \tilde{B}_u(t) u_0(t) dt = \mathcal{B}(u)$$

Therefore,

$$\int_0^{\vartheta} \gamma_u(t) u^T(t) G_u(t) u(t) dt + \mathcal{B}^T(u) F_1 \mathcal{B}(u) \geq$$

$$\geq \lambda^T \bar{P}_u \lambda + \lambda^T \bar{P}_u F_1 \bar{P}_u \lambda = \lambda^T(\bar{P}_u + \bar{P}_u F_1 \bar{P}_u)\lambda \geq 0$$

since the matrix $\bar{P}_u + \bar{P}_u F_1 \bar{P}_u$ is positive-semidefinite.

Lemma 1.2. Let the matrix $\bar{P}_u + \bar{P}_u F_1 \bar{P}_u$ be positive-semidefinite. Then for any function $\upsilon_0 \in L_q^2$ the functional $u \mapsto J(u, \upsilon_0)$ is convex on the set of admissible programmed strategies U.

Proof. Let $u_1, u_2 \in U$, $\lambda \in [0; 1]$ be given arbitrarily. Put

$$u_0 = \lambda u_1 + (1 - \lambda) u_2, \quad \Delta u = u_2 - u_1$$

$$z_i(t) = z_0 + \int_0^t (\tilde{B}_u(\tau) u_i(\tau) + \tilde{B}_\upsilon(\tau) \upsilon_0(\tau)) d\tau, \quad i \in \{1, 2\}$$

$$z = \lambda z_1 + (1 - \lambda) z_2, \quad \Delta z = z_2 - z_1$$

$$\delta J = \lambda J(u_1, \upsilon_0) + (1 - \lambda) J(u_2, \upsilon_0) - J(u_0, \upsilon_0)$$

We have to show that $\delta J \geq 0$.

Note that

$$u_1 = u_0 - (1-\lambda)\Delta u, \quad u_2 = u_0 + \lambda\Delta u, \quad z_1 = z - (1-\lambda)\Delta z, \quad z_2 = z + \lambda\Delta z$$

$$\Delta z(t) = \int_0^t \tilde{B}_u(\tau)\Delta u(\tau)d\tau, \quad \delta J = \delta_1 J + \delta_2 J$$

where

$$\delta_1 J = \frac{1}{2}(\lambda z_1^T(\vartheta)F_1 z_1(\vartheta) + (1-\lambda)z_2^T(\vartheta)F_1 z_2(\vartheta) - z^T(\vartheta)F_1 z(\vartheta))$$

$$\delta_2 J = \int_0^\vartheta (\beta_u(t, u_0(t)) - \lambda\beta_u(t, u_1(t)) - (1-\lambda)\beta_u(t, u_2(t)))dt$$

The expression for $\delta_1 J$ can be simplified as follows:

$$\begin{aligned}\delta_1 J &= \frac{1}{2}(\lambda(z(\vartheta) - (1-\lambda)\Delta z(\vartheta))^T F_1(z(\vartheta) - (1-\lambda)\Delta z(\vartheta)) + \\ &+ (1-\lambda)(z(\vartheta) + \lambda\Delta z(\vartheta))^T F_1(z(\vartheta) + \lambda\Delta z(\vartheta)) - z^T(\vartheta)F_1 z(\vartheta)) = \\ &= \frac{\lambda(1-\lambda)}{2}(\Delta z(\vartheta))^T F_1 \Delta z(\vartheta)\end{aligned} \tag{1.10}$$

We shall show that for any $t \in [0; \vartheta]$

$$\begin{aligned}&\beta_u(t, u_0(t)) - \lambda\beta_u(t, u_1(t)) - (1-\lambda)\beta_u(t, u_2(t)) \geq \\ &\geq \frac{\lambda(1-\lambda)}{2}\gamma_u(t)(\Delta u(t))^T G_u(t)\Delta u(t)\end{aligned} \tag{1.11}$$

Indeed, fit $t \in [0; \vartheta]$ and put

$$\bar{u}_1 = u_1(t), \quad \bar{u}_2 = u_2(t), \quad \bar{u}_0 = u_0(t), \quad \overline{\Delta u} = \bar{u}_2 - \bar{u}_1$$

$$\varphi(\tau) = \beta_u(t, \bar{u}_0 + \tau\overline{\Delta u}), \text{ where } \tau \in [-(1-\lambda); \lambda]$$

Then

$$\beta_u(t, u_0(t)) - \lambda\beta_u(t, u_1(t)) - (1-\lambda)\beta_u(t, u_2(t)) = \varphi(0) - \lambda\varphi(-(1-\lambda)) - (1-\lambda)\varphi(\lambda)$$

The function φ is continuous in the interval $[-(1-\lambda); \lambda]$ and infinitely differentiable in the interval $(-(1-\lambda); \lambda)$. Therefore, using Taylor's formula with the Lagrange remainder term, we conclude that numbers $\xi_1, \xi_2 \in (-(1-\lambda); \lambda)$ exist such that

$$\begin{aligned}&\varphi(0) - \lambda\varphi(-(1-\lambda)) - (1-\lambda)\varphi(\lambda) = \\ &= \varphi(0) - \lambda\left(\varphi(0) - (1-\lambda)\varphi'(0) + \frac{(1-\lambda)^2}{2}\varphi''(\xi_1)\right) - \\ &- (1-\lambda)\left(\varphi(0) + \lambda\varphi'(0) + \frac{\lambda^2}{2}\varphi''(\xi_2)\right) = \\ &= -\frac{\lambda(1-\lambda)}{2}((1-\lambda)\varphi''(\xi_1) + \lambda\varphi''(\xi_2))\end{aligned}$$

Since

$$\varphi(\tau) = \beta_u(t, \bar{u}_0 + \tau\overline{\Delta u}) = \gamma_u(t)\sqrt{1 - (\bar{u}_0 + \tau\overline{\Delta u})^T G_u(t)(\bar{u}_0 + \tau\overline{\Delta u})}$$

it follows that

$$\varphi''(\tau) = -\frac{\gamma_u^2(t)(\overline{\Delta u})^T G_u(t)\overline{\Delta u}}{\beta_u(t, \bar{u}_0 + \tau\overline{\Delta u})} - \frac{\gamma_u^4(t)((\overline{\Delta u})^T G_u(t)(\bar{u}_0 + \tau\overline{\Delta u}))^2}{\beta_u^3(t, \bar{u}_0 + \tau\overline{\Delta u})} \leq -\gamma_u(t)(\overline{\Delta u})^T G_u(t)\overline{\Delta u}$$

Consequently

$$\varphi(0) - \lambda\varphi(-(1-\lambda)) - (1-\lambda)\varphi(\lambda) \geq \frac{\lambda(1-\lambda)}{2}\gamma_u(t)(\overline{\Delta u})^T G_u(t)\overline{\Delta u}$$

that is, inequality (1.11) holds.

It follows from condition (1.11) that

$$\delta_2 J \geq \frac{\lambda(1-\lambda)}{2}\int_0^{\vartheta} \gamma_u(t)\Delta u(t))^T G_u(t)\Delta u(t)dt$$

Therefore, according to Eq. (1.10)

$$\delta J \geq \frac{\lambda(1-\lambda)}{2}\left(\int_0^{\vartheta} \gamma_u(t)(\Delta u(t))^T G_u(t)\Delta u(t)dt + (\Delta z(\vartheta))^T F_1 \Delta z(\vartheta)\right)$$

Hence, taking into account that $\Delta z(\vartheta) = \mathcal{B}(\Delta u)$, we infer the truth of the required inequality $\delta J \geq 0$ from Lemma 1.1.

Lemma 1.3. The sets of admissible controls U and V are convex compact sets in the weak topologies of the spaces L_p^2 and L_q^2, respectively.

Proof. The convexity of the sets U and V follows from the fact that the matrices $G_u(t)$ and $G_v(t)$ are positive-definite and from formula (1.4). We shall show that U is a compact set in the weak topology of the space L_p^2. The compactness of V in the weak topology of L_p^2 is proved in analogous fashion.

Since the matrix $G_u(t)$ is piecewise continuous in t and positive-definite for $t \in [0; \vartheta]$, a number $\varepsilon > 0$ exists such that $u^T G_u(t)u \geq \varepsilon u^T u$ for any $t \in [0; \vartheta]$, $u \in \mathbb{R}^p$. Consequently, for any function $u \in U$,

$$\|u\|^2_{L_p^2} \leq \frac{1}{\varepsilon}\int_0^{\vartheta} u^T G_u(t)udt \leq \frac{\vartheta}{\varepsilon} < +\infty$$

Hence the set U is bounded in the norm of the space L_p^2. This implies [4] that the set U is precompact in the weak topology of L_p^2.

It is easy to see that the set of admissible programmed strategies U is closed in the strongly topology of the space L_p^2. Hence, by virtue of the convexity of U, it follows [5] that U is closed in the weak topology of L_p^2. Since U is precompact and closed, it is compact.

Lemma 1.4. For any function $\upsilon_0 \in L_q^2$, the functional $u \mapsto J(u, \upsilon_0)$ is continuous in the set U in the sense of the strong topology of the space L_p^2.

Proof. Let $\{u_n\}$ be a sequence of elements of the set U that is strongly convergent in L_p^2 to an element $\bar{u} \in U$. Since $\mathcal{B}(u_n) \to \mathcal{B}(\bar{u})$ as $n \to \infty$, the terminal part of the performance index is convergent. It will therefore suffice to prove the convergence of the integral part of the functional, i.e. to show that as $n \to \infty$

$$\int_0^{\vartheta} a_n(t)dt \to \int_0^{\vartheta} \bar{a}(t)dt \tag{1.12}$$

where

$$a_n(t) = \gamma_u(t)\sqrt{1 - u_n^T(t)G_u(t)u_n(t)}, \quad \bar{a}(t) = \gamma_u(t)\sqrt{1 - \bar{u}^T(t)G_u(t)\bar{u}(t)}$$

Put

$$\Delta_n = \int_0^{\vartheta} |a_n(t) - \bar{a}(t)|dt, \quad \Delta_n^{(2)} = \int_0^{\vartheta} |a_n^2(t) - \bar{a}^2(t)|dt$$

Since $\{u_n\}$ converges to $\bar{u}$ and the functions $\gamma_u(t)$ and $G_u(t)$ are bounded, it follows that $\Delta_n^{(2)} \to 0$ as $n \to \infty$. Hence, by the inequality $|a_n(t) - \bar{a}(t)| \leq \sqrt{|a_n^2(t) - \bar{a}^2(t)|}$, which holds for the non-negative numbers $a_n(t)$, $\bar{a}(t)$, it follows that

$$\Delta_n \leq \int_0^{\vartheta} \sqrt{|a_n^2(t) - \bar{a}^2(t)|}dt \leq \sqrt{\vartheta\Delta_n^{(2)}} \to 0 \quad \text{as} \quad n \to \infty$$

that is, condition (1.2) is satisfied.

Lemma 1.5. Let the matrix $\bar{P}_u + \bar{P}_u F_1 \bar{P}_u$ be positive-semidefinite. Then for any function $\upsilon_0 \in L_q^2$ the functional $u \mapsto J(u, \upsilon_0)$ is lower semicontinuous on the set U in the weak topology of the space L_p^2.

Proof. Let $\{u_i\}$ be a sequence of elements of U converging weakly in L_p^2 to an element $\bar{u} \in L_p^2$. We have to prove that

$$J(\bar{u}, \upsilon_0) \le \underline{\lim}_{i \to \infty} J(u_i, \upsilon_0)$$

Since $\{u_i\}$ is weakly convergent to $\bar{u}$, sequence $\{\tilde{u}_n\}$ exists which is weakly convergent to $\bar{u}$ and is such that each element of $\{\tilde{u}_n\}$ is a convex combination of a finite number of elements of $\{u_i\}$. By Lemma 1.4, $J(\tilde{u}_n, \upsilon_0) \in J(\bar{u}, \upsilon_0)$ as $n \to \infty$.

By Lemma 1.2, the functional $u \mapsto J(u, \upsilon_0)$ is convex. Therefore

$$J(\bar{u}, \upsilon_0) = \lim_{n \to \infty} J(\tilde{u}_n, \upsilon_0) \le \underline{\lim}_{i \to \infty} J(u_i, \upsilon_0)$$

Theorem 1.1. Let the matrices $\bar{P}_u + \bar{P}_u F_1 \bar{P}_u$ and $\bar{P}_\upsilon - \bar{P}_\upsilon F_1 \bar{P}_\upsilon$ be positive-semidefinite. Then the differential game (1.1)–(1.3) has a saddle point in the class of programmed strategies:

$$\exists \hat{u} \in U, \quad \hat{\upsilon} \in V: \forall u \in U, \quad \upsilon \in V \quad J(\hat{u}, \upsilon) \le J(\hat{u}, \hat{\upsilon}) \le J(u, \hat{\upsilon})$$

Proof. It follows from Lemmas 1.1 and 1.5 that the functional $u \mapsto J(u, \upsilon)$ is convex and lower semicontinuous on the set U for any $\upsilon \in V$. One proves in similar fashion that the functional $\upsilon \mapsto J(u, \upsilon)$ is concave and upper semicontinuous on the set V for any $u \in U$. By Lemma 1.3, the sets U and V are convex and compact. Here, semicontinuity and compactness are understood in the sense of the weak topology of the space L_p^2 and L_q^2, respectively. The required assertion now follows from von Neumann's Theorem [6].

Using the matrix-valued function $P_u(t)$ and $P_\upsilon(t)$, defined in formula (1.9), we defined the following scalar functions for any $t \in [0; \vartheta]$; $\psi \in \mathbb{R}^m$

$$\sigma_u(t, \psi) = \sqrt{\gamma_u^2(t) + \psi^T P_u(t) \psi}, \quad \sigma_\upsilon(t, \psi) = \sqrt{\gamma_\upsilon^2(t) + \psi^T P_\upsilon(t) \psi} \tag{1.13}$$

and a vector-valued function

$$M(\psi) = F_1 z_0 - F_1 \int_0^\vartheta \frac{P_u(t)\psi}{\sigma_u(t, \psi)} dt + F_1 \int_0^\vartheta \frac{P_\upsilon(t)\psi}{\sigma_\upsilon(t, \psi)} dt \tag{1.14}$$

Theorem 1.2. Let the matrices $\bar{P}_u + \bar{P}_u F_1 \bar{P}_u$ and $\bar{P}_\upsilon - \bar{P}_\upsilon F_1 \bar{P}_\upsilon$ be positive-semidefinite. Let the vector ψ be a solution of the equation $\psi = M(\psi)$. Then the players' optimal programmed strategies are defined by the formulae

$$\hat{u}(t) = -\frac{G_u^{-1}(t) B_u^T(t) \Phi^T(t) \psi}{\sigma_u(t, \psi)}, \quad \hat{\upsilon}(t) = \frac{G_\upsilon^{-1}(t) B_\upsilon^T(t) \Phi^T(t) \psi}{\sigma_\upsilon(t, \psi)} \tag{1.15}$$

The value function of the game (its optimal guaranteed result) is defined by the formula

$$J(\hat{u}, \hat{\upsilon}) = \psi^T z_0 - \frac{1}{2} \psi^T F_1 \psi - \int_0^\vartheta (\sigma_u(t, \psi) - \sigma_\upsilon(t, \psi)) dt \tag{1.16}$$

Proof. Let the functions $\hat{u}(t)$ and $\hat{\upsilon}(t)$ be defined formulae (1.15). We shall show that $\hat{u}(t)$ and $\hat{\upsilon}(t)$ are optimal programmed strategies for the players.

It follows from formulae (1.9) and (1.15) that

$$\hat{u}^T(t) G_u(t) \hat{u}(t) = \frac{\psi^T \tilde{B}_u(t) G_u^{-1}(t) \tilde{B}_u^T(t) \psi}{\sigma_u^2(t, \psi)} = \frac{\psi^T P_u(t) \psi}{\sigma_u^2(t, \psi)}$$

Therefore, by Eq. (1.13)

$$\sqrt{1-\hat{u}^T(t)G_u(t)\hat{u}(t)} = \frac{\gamma_u(t)}{\sigma_u(t,\psi)} \tag{1.17}$$

For any vector $u \in \mathbb{R}^p$ such that $u^T G_u(t)u < 1$, the column vector of partial derivatives of the function $\beta_u(t, u)$ (defined by the first equality of (1.13)) with respect to the terms of the components of the vector u is equal to

$$\frac{\partial \beta_u(t,u)}{\partial u} = -\frac{\gamma_u(t)G_u(t)u}{\sqrt{1-u^T G_u(t)u}}$$

Hence, by formulae (1.5) and (1.17), it follows that

$$\frac{\partial \beta_u(t,\hat{u}(t))}{\partial u} = \tilde{B}_u^T(t)\psi \tag{1.18}$$

Let $\Delta u \in L_p^2$. Put

$$\hat{z}(t) = z_0 + \int_0^t (\tilde{B}_u(\tau)\hat{u}(\tau) + \tilde{B}_\upsilon(\tau)\hat{\upsilon}(\tau))d\tau, \quad \Delta z(t) = \int_0^t \tilde{B}_u(\tau)\Delta u(\tau)d\tau \tag{1.19}$$

It follows from formula (1.8) that, as $\tau \to 0$

$$J(\hat{u}+\tau\Delta u, \hat{\upsilon}) - J(\hat{u}, \hat{\upsilon}) =$$

$$= \tau\left(\hat{z}^T(\vartheta)F_1\Delta z(\vartheta) - \int_0^\vartheta \left(\frac{\partial \beta_u(t,\hat{u}(t))}{\partial u}\right)^T \Delta u(t)dt\right) + o(\tau)$$

Therefore the first variation of the functional $u \mapsto J(u, \hat{\upsilon})$ at the point $\hat{u}$ is equal to

$$\delta_u J(\Delta u) = \left(\hat{z}^T(\vartheta)F_1\int_0^\vartheta \tilde{B}_u(t)\Delta u(t)dt\right) - \int_0^\vartheta \left(\frac{\partial \beta_u(t,\hat{u}(t))}{\partial u}\right)^T \Delta u(t)dt$$

Hence, by formula (1.18), it follows that

$$\delta_u J(\Delta u) = \int_0^\vartheta (\hat{z}^T(\vartheta)F_1 - \psi^T)\tilde{B}_u(t)\Delta u(t)dt \tag{1.20}$$

Formulae (1.9) and (1.15) imply the equality

$$\tilde{B}_u(t)\hat{u}(t) = -\frac{P_u(t)\psi}{\sigma_u(t,\psi)} \tag{1.21}$$

Hence, by formulae (1.14) and (1.19), it follows that $\psi = M(\psi) = F_1\hat{z}(\vartheta)$. Therefore, by (1.20), we infer that the first variation of the functional $u \mapsto J(u, \hat{\upsilon})$ vanishes at the point $\hat{u}$. Taking into account that, as proved in Lemma 1.2, the functional $u \mapsto J(u, \hat{\upsilon})$ is convex, we obtain the inequality $J(\hat{u}, \hat{\upsilon}) \le J(u, \hat{\upsilon})$ for any function $u \in U$.

Similarly, for any function $\upsilon \in V$, we have an inequality $J(\hat{u}, \upsilon) \le J(\hat{u}, \hat{\upsilon})$. Consequently, the pair of programmed strategies $\hat{u}$, $\hat{\upsilon}$ constitutes a saddle point of the differential game, that is, a pair of optimal strategies of the players.

Since $\psi = F_1\hat{z}(\vartheta)$ and $F_1F_1 = E_m$ it follows that $\hat{z}(\vartheta) = F_1\psi$. Hence $\hat{z}^T(\vartheta)F_1\hat{z}(\vartheta) = \psi^T F_1\psi$. Therefore, by (1.8), we have

$$J(\hat{u}, \hat{\upsilon}) + \frac{1}{2}\psi^T F_1\psi = \psi^T\hat{z}(\vartheta) + \int_0^\vartheta (-\beta_u(t,\hat{u}(t)) + \beta_\upsilon(t,\hat{\upsilon}(t)))dt \tag{1.22}$$

It follows from Eq. (1.17) that

$$\beta_u(t, \hat{u}(t)) = \frac{\gamma_u^2(t)}{\sigma_u(t, \psi)}$$

Therefore, by formulae (1.19), (1.21) and (1.22), we obtain

$$J(\hat{u}, \hat{v}) + \frac{1}{2}\psi^T F_1 \psi =$$

$$= \psi^T z_0 + \int_0^{\vartheta} (\psi^T(\tilde{B}_u(t)\hat{u}(t) + \tilde{B}_v(t)\hat{v}(t)) - \beta_u(t, \hat{u}(t)) + \beta_v(t, \hat{v}(t)))dt =$$

$$= \psi^T z_0 + \int_0^{\vartheta}\left(-\frac{\psi^T P_u(t)\psi}{\sigma_u(t, \psi)} + \frac{\psi^T P_v(t)\psi}{\sigma_v(t, \psi)} - \frac{\gamma_u^2(t)}{\sigma_u(t, \psi)} + \frac{\gamma_v^2(t)}{\sigma_v(t, \psi)}\right)dt$$

Hence, by (1.13), we obtain equality (1.16).

Remark 1.1. Theorem 1.2 yields the players' optimal strategies as explicit expressions in terms of the vector of conjugate variables ψ. Therefore, in order to determine the optimal strategies, it will suffice to evaluate the vector ψ, having solved the equation $\psi = M(\psi)$. The vector ψ has the same meaning as the vector of conjugate variables in Pontryagin's Maximum Principle. In this case, the vector ψ is independent of time, since the differential game has the simple dynamics (1.7).

In the next two sections, we shall consider methods of solving the equation $\psi = M(\psi)$ and investigate the convergence of these methods. The results will imply the existence of a solution of the equation $\psi = M(\psi)$.

2. COMPUTATION OF THE VECTOR OF CONJUGATE VARIABLES BY THE SIMPLY ITERATION METHOD

Using the function $\sigma_u(t, \psi)$ defined by formula (1.13), we define the function

$$\varrho_u(\psi) = \int_0^{\vartheta} \sigma_u(t, \psi)dt \tag{2.1}$$

Let $D_u(\psi)$ denote the gradient of the function $\varrho_u(\psi)$:

$$D_u(\psi) = \int_0^{\vartheta} \frac{P_u(t)\psi}{\sigma_u(t, \psi)}dt \tag{2.2}$$

A similar definition yields the function $\varrho_v(\psi)$, whose gradient we will denote by $D_v(\psi)$.

It follows from formula (1.14) that for any vector $\psi \in \mathbb{R}^m$,

$$M(\psi) = F_1(z_0 - D_u(\psi) + D_v(\psi)) \tag{2.3}$$

Let $H_u(\psi)$ and $H_v(\psi)$ denote the matrices of second derivates (Hessians) of the functions $\varrho_u(\psi)$ and $\varrho_v(\psi)$:

$$H_u(\psi) = \int_0^{\vartheta} \frac{P_u(t)}{\sigma_u(t, \psi)}dt - \int_0^{\vartheta} \frac{P_u(t)\psi\psi^T P_u(t)}{\sigma_u^3(t, \psi)}dt \tag{2.4}$$

and an analogous formula holds for $H_v(\psi)$.

Lemma 2.1. For any vector $\psi \in \mathbb{R}^m$, the matrices $H_u(\psi)$, $H_v(\psi)$, $\bar{P}_u - H_u(\psi)$ and $\bar{P}_v - H_v(\psi)$ are positive-semidefinite.

Proof. We shall show that for any vector $y \in \mathbb{R}^m$,

$$0 \le y^T H_u(\psi) y \le y^T \bar{P}_u y \tag{2.5}$$

Indeed, by Eqs (1.13) and (2.4),

$$y^T H_u(\psi) y = \int_0^{\vartheta} \frac{(y^T P_u(t) y)(\gamma_u^2(t) + \psi^T P_u(t)\psi) - (y^T P_u(t)\psi)^2}{\sigma_u^3(t,\psi)} dt \le$$

$$\le \int_0^{\vartheta} \frac{y^T P_u(t) y}{\gamma_u(t)} dt = y^T \bar{P}_u y$$

In addition, the Cauchy–Bunyakovskii inequality

$$(y^T P_u(t) y)(\psi^T P_u(t)\psi) \ge (y^T P_u(t)\psi)^2$$

implies the inequality $0 \le y^T H_u(\psi) y$. This proves condition (2.5). This condition in turn implies that the matrices $H_u(\psi)$ and $\bar{P}_u - H_u(\psi)$ are positive-semidefinite. That the matrices $H_\upsilon(\psi)$ and $\bar{P}_\upsilon - H_\upsilon(\psi)$ are positive-semidefinite can be proved in similar fashion.

Put

$$\alpha = \|\bar{P}_u\| + \|\bar{P}_\upsilon\|.$$

Lemma 2.2 Suppose

$$\alpha < 1 \tag{2.6}$$

Then the mapping $\psi \mapsto M(\psi)$ (see (1.14)) is contractive with coefficient α.

Proof. We use Lemma 2.1. Since the matrices $\bar{P}_u$ and $H_u(\psi)$ are symmetric and positive-semidefinite, it follows that

$$\|\bar{P}_u\| = \max_{y \in \mathbb{R}^m:\, |y| = 1} y^T \bar{P}_u y, \quad \|H_u(\psi)\| = \max_{y \in \mathbb{R}^m:\, |y| = 1} y^T H_u(\psi) y$$

Hence, since the matrix $\bar{P}_u - H_u(\psi)$ is positive-semidefinite, we obtain the inequality $\|H_u(\psi)\| \le \|\bar{P}_u\|$. A similar arguments yields $\|H_\upsilon(\psi)\| \le \|\bar{P}_\upsilon\|$.

Since the Jacobian of the mapping $\psi \mapsto M(\psi)$ is $D_M(\psi) = -F_1 H_u(\psi) + F_1 H_\upsilon(\psi)$ $\|F_1\| = 1$, it follows by condition (2.6) that

$$\|D_M(\psi)\| \le \|H_u(\psi)\| + \|H_\upsilon(\psi)\| \le \alpha < 1$$

Hence $\psi \mapsto M(\psi)$ is a contractive mapping with coefficient α.

Lemma 2.2 implies the following theorem.

Theorem 2.1. If condition (2.6) holds, the equation $\psi = M(\psi)$ has a unique solution $\psi \in \mathbb{R}^m$. This solution may be found by simple iterations:

$$\psi_{k+1} = M(\psi_k), \quad \psi = \lim_{k \to \infty} \psi_k \tag{2.7}$$

The simple iteration method is convergent for any initial approximation $\psi_0 \in \mathbb{R}^m$ at a linear rate:

$$|\psi_k - \psi| \le \alpha^k |\psi_0 - \psi| \tag{2.8}$$

3. COMPUTATION OF THE VECTOR OF CONJUGATE VARIABLES BY NEWTON'S METHOD

It follows from the relations (2.3) and $F_1 F_1 = E_m$ that the equation $\psi = M(\psi)$ is equivalent to the equation $D(\psi) = z_0$, where

$$D(\psi) = F_1\psi + D_u(\psi) - D_\upsilon(\psi) \tag{3.1}$$

Remark 3.1. Since the values of the vector-valued functions $D_u(\psi)$ and $D_\upsilon(\psi)$ equal the gradients of the scalar functions $\varrho_u(\psi)$ and $\varrho_\upsilon(\psi)$, it follows that the value of the vector-valued function $D(\psi)$ equals the gradient of the scalar function

$$\varrho(\psi) = \frac{1}{2}\psi^T F_1 \psi + \varrho_u(\psi) - \varrho_\upsilon(\psi) \tag{3.2}$$

Therefore, the equation $D(\psi) = z_0$ is equivalent to the statement that the vector of conjugate variables φ is a stationary point of the function $\psi \mapsto \varrho(\psi) - \psi^T z_0$.

Let $H(\psi)$ denote the Hessian of the function $\varrho(\psi)$. Since the Hessians of the functions $\varrho_u(\psi)$ and $\varrho_\upsilon(\psi)$ are equal to $H_u(\psi)$ and $H_\upsilon(\psi)$, respectively, it follows that

$$H(\psi) = F_1 + H_u(\psi) - H_\upsilon(\psi) \tag{3.3}$$

Let us partition of the matrices $\bar{P}_u$ and $\bar{P}_\upsilon$ into blocks:

$$\bar{P}_u = \begin{Vmatrix} P^u_{11} & P^u_{12} \\ (P^u_{12})^T & P^u_{22} \end{Vmatrix}, \quad \bar{P}_\upsilon = \begin{Vmatrix} P^\upsilon_{11} & P^\upsilon_{12} \\ (P^\upsilon_{12})^T & P^\upsilon_{22} \end{Vmatrix} \tag{3.4}$$

where $P^u_{11}, P^\upsilon_{11} \in \mathbb{R}^{r\times r}$, $P^u_{12}, P^\upsilon_{12} \in \mathbb{R}^{r\times s}$, $P^u_{22}, P^\upsilon_{22} \in \mathbb{R}^{s\times s}$, and the numbers r and s are defined by formula (1.5).

Lemma 3.1. Let

$$\|P^u_{22}\| < 1, \quad \|P^\upsilon_{11}\| < 1 \tag{3.5}$$

Then the function $\varrho(\psi)$ is convex in ψ_1 and concave in ψ_2, where

$$\psi = \begin{Vmatrix} \psi_1 \\ \psi_2 \end{Vmatrix}, \quad \psi_1 \in \mathbb{R}^r, \quad \psi_2 \in \mathbb{R}^s$$

Proof. Partition the matrices $H(\psi)$, $H_u(\psi)$ and $H_\upsilon(\psi)$ into blocks as follows:

$$H(\psi) = \begin{Vmatrix} H_{11} & H_{12} \\ H^T_{12} & H_{22} \end{Vmatrix}$$

$$H_u(\psi) = \begin{Vmatrix} H^u_{11} & H^u_{12} \\ (H^u_{12})^T & H^u_{22} \end{Vmatrix}, \quad H_\upsilon(\psi) = \begin{Vmatrix} H^\upsilon_{11} & H^\upsilon_{12} \\ (H^\upsilon_{12})^T & H^\upsilon_{22} \end{Vmatrix}$$

$$H_{11}, H^u_{11}, H^\upsilon_{11} \in \mathbb{R}^{r\times r}, \quad H_{12}, H^u_{12}, H^\upsilon_{12} \in \mathbb{R}^{r\times s}, \quad H_{22}, H^u_{22}, H^\upsilon_{22} \in \mathbb{R}^{s\times s}$$

By Lemma 2.1, the matrices H^u_{11}, H^υ_{22}, $P^\upsilon_{11} - H^\upsilon_{11}$ and $P^u_{22} - H^u_{22}$ are positive-semidefinite. Hence, by formula (3.3) and the inequality $\|P^\upsilon_{11}\| < 1$, the Hessian of the function ϱ with respect to the components of the vector ψ_1, which equals

$$H_{11} = E_r + H^u_{11} - H^\upsilon_{11} = (E_r - P^\upsilon_{11}) + (P^\upsilon_{11} - H^\upsilon_{11}) + H^u_{11}$$

is positive-definite. Consequently, ϱ is a convex function in ψ_1. The proof that ϱ is a concave function in ψ_2 is similar.

Remark 3.2. It follows from Lemma 3.1 and Remark 3.1 that the vector of conjugate variables ψ is a saddle point of the function $\psi \mapsto \varrho(\psi) - \psi^T z_0$.

It will be shown below that conditions (3.5) imply invertibility of the matrix $H(\psi)$. Under the same conditions, an estimate will be derived for the norm of the inverse matrix of $H(\psi)$, which will be needed for the rigorous proof that the algorithm for computing the vector of conjugate variables ψ is convergent. To that end we will need the following lemma, which belongs to linear algebra.

Lemma 3.2. Suppose we are given a number $\mu > 0$ and a matrix

$$A = \begin{Vmatrix} A_{11} & A_{12} \\ A^T_{12} & A_{22} \end{Vmatrix}$$

where $A_{11} \in \mathbb{R}^{r\times r}, A_{12} \in \mathbb{R}^{r\times s}, A_{22} \in \mathbb{R}^{s\times s}$. Let the matrix A be symmetric, the matrix $B = A_{11} - \mu E_r$ positive-semidefinite, and the matrix $C = A_{22} + \mu E_s$ negative-semidefinite. Then the matrix A is invertible and $\|A^{-1}\| \le 1/\mu$.

Proof. Given an arbitrary vector $x \in \mathbb{R}^{r+s}$, partition it into two vectors

$$x = \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix}, \quad \text{where} \quad x_1 \in \mathbb{R}^r, \quad x_2 \in \mathbb{R}^s$$

Then

$$Ax = y = \begin{Vmatrix} y_1 \\ y_2 \end{Vmatrix}, \quad \text{where} \quad y_1 = A_{11}x_1 + A_{12}x_2, \quad y_2 = A_{12}^T x_1 + A_{22}x_2$$

Since $A_{11} = B + \mu E_r, A_{22} = C - \mu E_s$, it follows that

$$y_1 = Bx_1 + A_{12}x_2 + \mu x_1, \quad y_2 = A_{12}^T x_1 + Cx_2 - \mu x_2$$

Consequently

$$\begin{aligned}
&|Ax|^2 = |y|^2 = |y_1|^2 + |y_2|^2 = \\
&= |Bx_1 + A_{12}x_2|^2 + 2\mu x_1^T(Bx_1 + A_{12}x_2) + \mu^2|x_1|^2 + \\
&+ |A_{12}^T x_1 + Cx_2|^2 - 2\mu x_2^T(A_{12}^T x_1 + Cx_2) + \mu^2|x_2|^2 \ge \\
&\ge 2\mu(x_1^T(Bx_1 + A_{12}x_2) - x_2^T(A_{12}^T x_1 + Cx_2)) + \mu^2|x|^2 = \\
&= 2\mu(x_1^T Bx_1 - x_2^T Cx_2) + \mu^2|x|^2 \ge \mu^2|x|^2
\end{aligned}$$

Therefore, $|Ax| \ge \mu|x|$. Hence the matrix A is invertible and $\|A^{-1}\| \le 1/\mu$.

Let the matrices P_{11}^{υ} and P_{22}^{u} be defined by formula (3.4). Define a number

$$\mu = \min\{1 - \|P_{22}^u\|, 1 - \|P_{11}^{\upsilon}\|\} \tag{3.6}$$

Lemma 3.3. Suppose condition (3.5) holds. Then for any $\psi \in \mathbb{R}^m$, the matrix $H(\psi)$ is invertible and

$$\|H^{-1}(\psi)\| \le 1/\mu$$

Proof. By Lemma 2.1, the matrices $H_{11}^u, H_{22}^{\upsilon}, P_{11}^{\upsilon} - H_{11}^{\upsilon}$ and $P_{22}^u - H_{22}^u$ are positive-semidefinite. It follows from the condition $\|P_{11}^{\upsilon}\| \le 1 - \mu$ that the matrix $(1-\mu)E_r - P_{11}^{\upsilon}$ is positive-semidefinite. Hence the matrix

$$H_{11} - \mu E_r = (1-\mu)E_r + H_{11}^u - H_{11}^{\upsilon} = ((1-\mu)E_r - P_{11}^{\upsilon}) + (P_{11}^{\upsilon} - H_{11}^{\upsilon}) + H_{11}^u$$

is also positive-semidefinite. Similarly, the matrix $H_{22} + \mu E_s$ is negative-semidefinite. Apply Lemma 3.2, we obtain the required assertion.

Lemma 3.4. The mapping $\psi \mapsto H(\psi)$ is Lipschitz continuous with constant

$$L_H = L_H^u + L_H^{\upsilon}; \quad L_H^u = 2\int_0^{\vartheta} \frac{\|P_u(t)\|^{3/2}}{\gamma_u^2(t)}, \quad L_H^{\upsilon} = 2\int_0^{\vartheta} \frac{\|P_{\upsilon}(t)\|^{3/2}}{\gamma_{\upsilon}^2(t)} \tag{3.7}$$

Proof. Fix arbitrary vectors $y, \Delta\psi \in \mathbb{R}^m$. If follows from formula (2.4) that

$$y^T H_u(\psi) y = \int_0^{\vartheta} \frac{y^T P_u(t) y}{\sigma_u(t,\psi)} dt - \int_0^{\vartheta} \frac{(y^T P_u(t)\psi)^2}{\sigma_u^3(t,\psi)} dt$$

Using formula (1.13), let us evaluate the derivative

$$d_1 = \frac{d}{dt} y^T H_u(\psi + t\Delta\psi) y$$

at the point $t = 0$

$$d_1 = -\int_0^{\vartheta}\frac{(y^TP_u(t)y)(\psi^TP_u(t)\Delta\psi)}{\sigma_u^3(t,\psi)}dt - $$
$$-\int_0^{\vartheta}\frac{2(y^TP_u(t)\psi)(y^TP_u(t)\Delta\psi)}{\sigma_u^3(t,\psi)}dt + \int_0^{\vartheta}\frac{3(y^TP_u(t)\psi)^2(\psi^TP_u(t)\Delta\psi)}{\sigma_u^5(t,\psi)}dt$$

By the Cauchy–Bunyakovskii inequality

$$|d_1| \le \int_0^{\vartheta}\frac{3(y^TP_u(t)y)\sqrt{\Delta\psi^TP_u(t)\Delta\psi}\sqrt{\psi^TP_u(t)\psi}}{\sigma_u^3(t,\psi)}dt +$$
$$+\int_0^{\vartheta}\frac{3(y^TP_u(t)y)\sqrt{\Delta\psi^TP_u(t)\Delta\psi}(\psi^TP_u(t)\psi)^{3/2}}{\sigma_u^5(t,\psi)}dt \le$$
$$\le 3C\int_0^{\vartheta}\frac{(y^TP_u(t)y)\sqrt{\Delta\psi^TP_u(t)\Delta\psi}}{\gamma_u^2(t)}dt \le 3C\left(\int_0^{\vartheta}\frac{\|P_u(t)\|^{3/2}}{\gamma_u^2(t)}dt\right)|y|^2|\Delta\psi|$$

where

$$C = \max_{x\in\mathbb{R}}\left(\frac{x}{(1+x^2)^{3/2}} + \frac{x^3}{(1+x^2)^{5/2}}\right) = 4\cdot 5^{-5/4} < \frac{2}{3}$$

Consequently, the mapping $\psi \mapsto H_u(\psi)$ is Lipschitz continuous with constant L_H^u. Similarly, the mapping $\psi \mapsto H_\upsilon(\psi)$ is Lipschitz continuous with constant L_H^υ.

Theorem 3.1. Suppose condition (3.5) holds and let $\psi_0 \in \mathbb{R}^m$ be a given vector. Suppose the sequence $\{\psi_k\}$ is defined by Newton's method,

$$\psi_{k+1} = \psi_k + (H(\psi_k))^{-1}(z_0 - D(\psi_k)) \tag{3.8}$$

The error in the solution of the equation $D(\psi) = z_0$ at step k of method (3.8) is defined as the number

$$\delta_k = \frac{L_H}{2\mu^2}|D(\psi_k) - z_0|$$

where the number μ and L_H are defined by formulae (3.6) and (3.7), respectively. Suppose the error of initial approximation is less than 1

$$\delta_0 < 1 \tag{3.9}$$

Then Newton's method (3.8) will converge to a solution of the equation $D(\psi) = z_0$ at a quadratic rate:

$$\delta_{k+1} \le \delta_k^2 \tag{3.10}$$

Proof. Since by Lemma 3.4 the matrix-valued function $H(\psi)$ is Lipschitz continuous with constant L_H, it follows from Taylor's formula that

$$|D(\psi_{k+1}) - D(\psi_k) - H(\psi_k)(\psi_{k+1} - \psi_k)| \le \frac{1}{2}L_H|\psi_{k+1} - \psi_k|^2.$$

From formula (3.8) we obtain $D(\psi_k) + H(\psi_k)(\psi_{k+1} - \psi_k) = z_0$. Therefore

$$|D(\psi_{k+1}) - z_0| \le \frac{1}{2}L_H|\psi_{k+1} - \psi_k|^2 \tag{3.11}$$

Since by Lemma 3.3 $\|H^{-1}(\psi)\| \le 1/\mu$, it follows by virtue of (3.8) that

$$|\psi_{k+1} - \psi_k| \le |D(\psi_k) - z_0|/\mu$$

Hence, by inequality (3.11), we obtain relation (3.10). It follows from (3.9) and (3.10) by induction that $\delta_k \le \delta_0^{2^k} \to 0$ as $k \to \infty$. The theorem is proved.

As is well known, if the initial approximation is insufficiently accurate, Newton's method may diverge. It is thus very important to establish an initial approximation from which Newton's method will converge. We will consider a possible method of obtaining an initial approximation which ensures that Newton's method will be convergent.

Lemma 3.5. Suppose condition (3.5) holds. Define a natural number N by the condition

$$N \ge 2|z_0|L_H/\mu^2 \tag{3.12}$$

Define

$$z_k = kz_0/N, \quad \psi_1^0 = 0, \quad \psi_{k+1}^0 = \psi_k^0 + (H(\psi_k^0))^{-1}(z_k - D(\psi_k^0)), \quad k \in \{1, \dots, N\} \tag{3.13}$$

Then the vector $\psi_0 = \psi_N^0$ satisfies inequality (3.9).

Proof. As in the proof of inequality (3.10), we will prove by induction that the following inequality holds for $k \in \{1, \dots, N\}$

$$|D(\psi_k^0) - z_k| \le 2|z_0|/N$$

Together with condition (3.12), this yields the desired inequality (3.9).

We will now consider the question of the mutual dependence of condition (3.5), which guarantees the convergence of Newton's method, and the condition that the matrices $\bar{P}_u + \bar{P}_u F_1 \bar{P}_u$ and $\bar{P}_\upsilon - \bar{P}_\upsilon F_1 \bar{P}_\upsilon$ be positive-semidefinite which, by Theorem 1.1, guarantees the existence of a saddle point of the differential game 91.1)–(1.3).

Lemma 3.6. Suppose we are given

$$F = \begin{Vmatrix} E_r & 0 \\ 0 & -E_s \end{Vmatrix}, \quad A = \begin{Vmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{Vmatrix}$$

where $A_{11} \in \mathbb{R}^{r \times r}, A_{12} \in \mathbb{R}^{r \times s}, A_{22} \in \mathbb{R}^{s \times s}$. Let A be a symmetric positive-semidefinite matrix. Then (1) the inequality $\|A_{22}\| \le 1$ implies that the matrix $A + AFA$ is positive-semidefinite, and (2) the converse is false.

Proof 1. Let $\|A_{22}\| \le 1$. We must show that the matrix $A + AFA$ is positive-semidefinite. Suppose the contrary: a vector $x_0 \in \mathbb{R}^{r+s}$: $x_0^T(A + AFA)x_0 < 0$ exists. Since A is symmetric and positive-semidefinite, a matrix $S \in \mathbb{R}^{m \times (r+s)}$ exists (where m is the rank of A) such that $A = S^T S$. Consequently, a vector $y_0 = Sx_0 \in \mathbb{R}^m$ exists such that $y_0^T(E_m + SFS^T)y_0 < 0$. Therefore,

$$\min_{y \in \mathbb{R}^m:\, |y| = 1} y^T(E_m + SFS^T)y < 0$$

Suppose the minimum is obtained for a vector y_1. Then a number λ exists such that y_1 is a stationary point of the Lagrange function

$$y \mapsto y^T(E_m + SFS^T)y - \lambda y^T y$$

that is

$$(E_m + SFS^T)y_1 = \lambda y_1 \tag{3.14}$$

Since

$$\min_{y \in \mathbb{R}^m:\, |y| = 1} y^T(E_m + SFS^T)y = y_1^T(E_m + SFS^T)y_1 = \lambda y_1^T y_1 < 0$$

it follows that $\lambda < 0$ and $y_1 \neq 0$.

It follows from Eq. (3.14) that $S^T(E_m + SFS^T)y_1 = \lambda S^T y_1$, and so $(S^T + AFS^T)y_1 = \lambda S^T y_1$, that is

$$(E_{r+s} + AF)z = \lambda z \tag{3.15}$$

where $z = S^T y_1 \in \mathbb{R}^{r+s}$. Since the columns of the matrix S^T are linearly independent and $y_1 \neq 0$, it follows that $z \neq 0$.

We write

$$z = \begin{Vmatrix} z_1 \\ z_2 \end{Vmatrix}, \quad \text{where} \quad z_1 \in \mathbb{R}^r, \quad r_2 \in \mathbb{R}^s$$

Then Eq. (3.15) becomes a system

$$z_1 + A_{11}z_1 - A_{12}z_2 - \lambda z_1 = 0$$

$$z_2 + A_{12}^T z_1 - A_{22}z_2 - \lambda z_2 = 0$$

Multiplying the first equation of the system on the left by z_1^T, the second equation by z_2^T and adding, we obtain the equation

$$z_1^T(A_{11} + (1-\lambda)E_r)z_1 + z_2^T((1-\lambda)E_s - A_{22})z_2 = 0 \tag{3.16}$$

Since $\lambda < 0$ and the matrix A_{11} is positive-semidefinite, it follows that the matrix $A_{11} + (1-\lambda)E_r$ is positive-definite. Since $\lambda < 0$ and $\|A_{22}\| \leq 1$, the matrix $(1-\lambda)E_s - A_{22}$ is positive-definite. Hence Eq. (3.16) contradicts the condition $z \neq 0$.

We will now show that the fact that the matrix $A + AFA$ is positive-semidefinite does not necessarily imply that $\|A_{22}\| \leq 1$. Let

$$A = \begin{Vmatrix} 5/2 & 2 \\ 2 & 2 \end{Vmatrix}, \quad F = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$$

Then the matrices A and

$$A + AFA = \begin{Vmatrix} 19/4 & 3 \\ 3 & 2 \end{Vmatrix}$$

are positive-definite, but $\|A_{22}\| = 2 > 1$.

Remark 3.3. Suppose the conditions of Theorem 3.1 are satisfied: $\|P_{22}^u\| < 1$ and $\|P_{11}^{\upsilon}\| < 1$. Then by Lemma 3.6 the matrix $\bar{P}_u + \bar{P}_u F_1 \bar{P}_u$ is positive-semidefinite. Similarly, the matrix $\bar{P}_{\upsilon} - \bar{P}_{\upsilon} F_1 \bar{P}_{\upsilon}$ is positive-semidefinite. Hence, by Theorems 1.1 and 1.2, the differential game has a saddle point, and formulae (1.15) and (1.16) for the optimal strategies and the guaranteed result are valid.

4. A DIFFERENTIAL GAME WITH PURELY GEOMETRICAL CONSTRAINTS ON THE PURSUER'S CONTROL

In this section, we will consider a differential game (DG) (1.1)–(1.3) such that the terminal term $x^T(\vartheta)Fx(\vartheta)/2$ of the performance index is a convex function, that is, the matrix F is positive-semidefinite. In that case player u, minimizing the performance index, will strive to reduce the length of the phase vector. Player υ will strive to increase the length of the phase vector. In that sense, we shall call player u the pursuer and player υ the evader.

We shall consider a DG for which $\gamma_u(t) = 0$. Such a DG will be denote by DG_0 and referred to as a DG with purely geometrical constraints on the pursuer's control.

A DG for which $\gamma_u(t) = \varepsilon \geq 0$ will be denote by DG_ε. For any admissible programmed controls $u \in U$, $\upsilon \in V$, the value of the performance index in DG_ε will be denoted by $J_\varepsilon(u, \upsilon)$.

An optimal programmed strategy for the pursuer in a DG_ε will be denoted by $\hat{u}_\varepsilon$ and called an ε-strategy for the pursuer in the DG_0.

Such as ε-strategy $\hat{u}_\varepsilon$ has two advantages over an exact optimal guaranteed strategy $\hat{u}_0$. First, an ε-strategy is a Lipschitz continuous function of time and the game parameters. Second, the algorithm that computes an ε-strategy is simple and effective from a computational standpoint. One should bear in mind that as $\varepsilon \to 0$ the Lipschitz constants of the ε-strategy tend to infinity, while the rate of convergence of the algorithms decreases. Accordingly, the parameter ε characterizing the accuracy of the ε-strategy should not be taken too small.

Theorem 4.1. Let the matrix F be positive-semidefinite and $||\bar{P}_\upsilon|| < 1$. Then the DG_0 has a saddle point $(\hat{u}_0, \hat{\upsilon}_0)$ in the class of programmed strategies. The value of the performance index guaranteed by the pursuer's ε-strategy is at most the exact value of the optimal guaranteed result plus $\varepsilon\vartheta$:

$$J_0(\hat{u}_\varepsilon, \upsilon) \le J_0(\hat{u}_0, \hat{\upsilon}_0) + \varepsilon\vartheta \quad \forall \upsilon \in V \tag{4.1}$$

and the pursuer's ε-strategy is defined by the formula

$$\hat{u}_\varepsilon(t) = -\frac{G_u^{-1}(t)B_u^T(t)\Phi^T(t)\psi}{\sqrt{\varepsilon^2 + \psi^T P_u(t)\psi}}$$

The vector of conjugate variables ψ is a solution of the equation $D(\psi) = z_0$, where

$$z_0 = \Phi(0)x_0, \quad D(\psi) = \psi + \int_0^\vartheta \frac{P_u(t)\psi}{\sqrt{\varepsilon^2 + \psi^T P_u(t)\psi}}dt - \int_0^\vartheta \frac{P_\upsilon(t)\psi}{\sqrt{\gamma_\upsilon^2(t) + \psi^T P_\upsilon(t)\psi}}dt$$

Proof. Since the matrix F is positive-semidefinite, it follows that for any strategy $\upsilon \in V$ the functional $u \mapsto J_0(u, \upsilon)$ is convex on the set U. Since the matrices $\bar{P}_\upsilon$ and $E_m - \bar{P}_\upsilon$ are positive-semidefinite, the same is true of the matrix $\bar{P}_\upsilon - \bar{P}_\upsilon F_1 \bar{P}_\upsilon$, where $F_1 = E_m$. Hence, as in the proof of Lemma 1.2, we infer that the functional $\upsilon \mapsto J_0(u, \upsilon)$ is concave on the set V for any strategy $u \in U$.

As in the proof of Lemma 1.5, it can be shown that the functional $J_0(u, \upsilon)$ is lower semicontinuous in u and upper semicontinuous in υ in the weak topology of the spaces L_p^2, L_q^2, respectively. Since by Lemma 1.3 the sets U and V are convex and compact in those topologies, Neumann's Theorem implies the existence of a saddle point $(\hat{u}_0, \hat{\upsilon}_0)$ of the functional J_0.

It follows from formulae (1.20) and (1.3) that for any admissible programmed strategies $u \in U$, $\upsilon \in V$.

$$J_\varepsilon(u, \upsilon) \le J_0(u, \upsilon) \le J_\varepsilon(u, \upsilon) + \varepsilon\vartheta$$

Therefore, for any admissible programmed strategy $\upsilon \in V$, we have the inequalities

$$J_0(\hat{u}_\varepsilon, \upsilon) \le J_\varepsilon(\hat{u}_\varepsilon, \upsilon) + \varepsilon\vartheta \le J_\varepsilon(\hat{u}_\varepsilon, \hat{\upsilon}_\varepsilon) + \varepsilon\vartheta$$

Hence, since moreover

$$J_\varepsilon(\hat{u}_\varepsilon, \hat{\upsilon}_\varepsilon) = \min_{u \in U}\max_{\upsilon \in V} J_\varepsilon(u, \upsilon) \le \min_{u \in U}\max_{\upsilon \in V} J_0(u, \upsilon) = J_0(\hat{u}_0, \hat{\upsilon}_0)$$

we obtain inequality (4.1). The expression for the ε-strategy follows from formula (1.5).

Applying Theorem 3.1 to a DG_ε, we obtain the following theorem, establishing the convergence of Newton's method for computing the vector ψ from which the pursuer's ε-strategy is determined.

Theorem 4.2. Assume that the conditions of Theorem 4.1 are satisfied. Let $\{\psi_n\}$ be a sequence defined by formula (3.8), with the initial approximation $\psi_0 = \psi_N^0$ defined by formula (3.13) with

$$N \ge \frac{2|z_0|L_H^\varepsilon}{\mu^2}, \quad \mu = 1 - \|\bar{P}_\upsilon\|, \quad L_H^\varepsilon = 2\int_0^\vartheta \left(\frac{\|P_u(t)\|^{3/2}}{\varepsilon^2} + \frac{\|P_\upsilon(t)\|^{3/2}}{\gamma_\upsilon^2(t)}\right)dt \tag{4.2}$$

Then the sequence $\{\psi_n\}$ converges to a solution of the equation $D(\psi) = z_0$ at a quadratic rate:

$$|D(\psi_{n+1}) - z_0| \le \frac{L_H^\varepsilon}{2\mu^2}|D(\psi_n) - z_0|^2$$

Remark 4.1. Since the matrix $P_u(t)$ is independent of $\gamma_u(t)$, formula (1.9) implies that $\|\bar{P}_u\| \sim 1/\varepsilon$. Therefore, for sufficiently small ε, inequality (2.6) will fail to hold and the simple iteration method described in Theorem 2.1 may diverge.

Remark 4.2. If the matrix F is positive-semidefinite and moreover $\|\bar{P}_\upsilon\| < 1$, then by Lemma 3.1 the function $\varrho(\psi)$ is convex. Therefore, by Remark 3.1, the vector of conjugate variables ψ is a minimum point of the function $\psi \mapsto \varrho(\psi) - \psi^T z_0$.

Remark 4.3. It follows from formula (4.2) that $N \sim 1/\varepsilon^2$ as $\varepsilon \to +0$. For small ε, therefore, the number N becomes very large. In that case the algorithm determining the initial approximation ψ_0 given by formula (3.13) may be very time-consuming. It may well be more effective to use well-known convex optimization algorithms to approximate the minimum point of the convex function $\psi \mapsto \varrho(\psi) - \psi^T z_0$. Experience in numerical computations shows that in many case Newton's method (3.8) converges from the initial approximation $\psi_0 = 0$. In such cases there is no longer any need to run the time-consuming algorithm (3.13).

5. DIFFERENTIAL GAMES WITHOUT GEOMETRICAL CONSTRAINTS ON THE PLAYERS' CONTROLS

We will now consider the limiting case of DG (1.1)–(1.3) with

$$G_\upsilon(t) = \varepsilon G_\upsilon^0(t), \quad \gamma_\upsilon(t) = 1/\varepsilon, \quad \varepsilon \to +0 \tag{5.1}$$

where $G_\upsilon^0(t)$ is a given symmetric positive-definite matrix which is a piecewise-continuous function of the time t.

Having in mid the limit relation

$$\beta_\upsilon(t, \upsilon) = \frac{1}{\varepsilon}\sqrt{1 - \upsilon^T G_\upsilon(t)\upsilon} = \frac{1}{\varepsilon} - \frac{1}{2}\upsilon^T G_\upsilon^0(t)\upsilon + o(1) \quad \text{as} \quad \varepsilon \to +0$$

let us say that the limiting case of DG (1.1)–(1.3) under conditions (5.1) is DG (1.1), (1.2), where

$$\beta_u(t, u) = \gamma_u(t)\sqrt{1 - u^T G_u(t)u} \quad \beta_\upsilon(t, \upsilon) = -\frac{1}{2}\upsilon^T G_\upsilon^0(t)\upsilon \tag{5.2}$$

We define a matrix-valued function

$$P_\upsilon^0(t) = \Phi(t)B_\upsilon(t)(G_\upsilon^0(t))^{-1}B_\upsilon^T(t)\Phi^T(t) \tag{5.3}$$

Then, in accordance with the notation (1.9),

$$P_u(t) = \Phi(t)B_u(t)G_u^{-1}(t)B_u^T(t)\Phi^T(t)$$

$$\bar{P}_u = \int_0^\vartheta \frac{1}{\gamma_u(t)}P_u(t)dt, \quad \bar{P}_\upsilon = \int_0^\vartheta P_\upsilon^0(t)dt$$

Repeating the arguments used to prove Theorems 1.1 and 1.2, we obtain the following theorem.

Theorem 5.1. Let the matrices $\bar{P}_u + \bar{P}_u F_1 \bar{P}_u$ and $\bar{P}_\upsilon - \bar{P}_\upsilon F_1 \bar{P}_\upsilon$ be positive-semidefinite. Then DG (1.1), (1.2), (5.2) has a saddle point in the class of programmed strategies. The players' optimal programmed strategies are defined by the formulae

$$\hat{u}(t) = -\frac{G_u^{-1}(t)B_u^T(t)\Phi^T(t)\psi}{\sigma_u(t, \psi)}, \quad \hat{\upsilon}(t) = (G_\upsilon^0(t))^{-1}B_\upsilon^T(t)\Phi^T(t)\psi \tag{5.4}$$

The vector of conjugate variables ψ is determined from the equation $D(\psi) = z_0$, where

$$D(\psi) = F_1\psi + \int_0^\vartheta \frac{P_u(t)\psi}{\sigma_u(t, \psi)}dt - \bar{P}_\upsilon\psi, \quad z_0 = \Phi(0)x_0$$

The value function of the DG (the optimal guaranteed result) is defined by the formula

$$J(\hat{u}, \hat{v}) = \psi^T z_0 - \frac{1}{2}\psi^T F_1 \psi - \int_0^{\vartheta} \sqrt{\sigma_u(t, \psi)}dt \tag{5.5}$$

The solution of the equation $D(\psi) = z_0$ may be found by a simple iterative method or by Newton's method, as was done in Theorems 2.1 and 3.1.

Similar reasoning may be applied to the limiting case of DG (1.1)–(1.3) under conditions (5.1) with υ replaced by u.

In the limiting case in which conditions (5.1) hold simultaneously with the same conditions for player u, DG (1.1)–(1.3) becomes the well-known linear-quadratic DG [1, p. 160].

6. AN EXAMPLE OF A DIFFERENTIAL GAME

As an example, we will consider two DGs of the form (1.1)–(1.3) with parameters $n = 4, p = q = 2$, $\vartheta = 12, F = E_4, G_u = G_\upsilon = E_2, \gamma_\upsilon(t) = 16$

$$x_0 = \begin{Vmatrix} 9 \\ 4 \\ 8 \\ 7 \end{Vmatrix}, \quad A(t) = \begin{Vmatrix} 0.1 & 0.9 & -0.1 & 0.1 \\ -1.0 & -0.2 & 0.2 & -0.3 \\ 0.3 & 0.1 & 0.1 & 1.5 \\ -0.1 & 0.2 & -1.4 & -0.3 \end{Vmatrix}$$

$$B_u(t) = \begin{Vmatrix} 1.0 & 0.2 \\ 0.3 & -0.3 \\ -0.1 & 0.9 \\ 0.5 & -0.2 \end{Vmatrix}, \quad B_\upsilon(t) = \begin{Vmatrix} 0.7 & 0.1 \\ 1.2 & 0.2 \\ 0.1 & -0.3 \\ -0.3 & 1.1 \end{Vmatrix}$$

For the first DG, $\gamma_u^1(t) = 5$, while for the second, $\gamma_u^2(t) = 0.5$.

Numerical computations yield the following values of the vectors of conjugate variables for the first and second DGs

$$\psi^1 = \begin{Vmatrix} -9.03 \\ -1.65 \\ 0.178 \\ 6.06 \end{Vmatrix}, \quad \psi^2 = \begin{Vmatrix} -7.23 \\ -1.44 \\ 0.376 \\ 4.78 \end{Vmatrix}$$

Figure 1 illustrates the hodographs of the optimal control vectors for players u and υ. Figure 2 shows projections of the trajectories of the phase vector $x(t)$ corresponding to optimal controls. The graphs for the first DG are the solid curves while those for the second are the dashed curves.

The graphs shown in the figures clearly demonstrate some properties of DGs with ellipsoidal penalties.The DGs considered are distinguished only by the penalty coefficient, which indicates the dependence of the performance index on player u's control. Since the penalty coefficient in the second DG is smaller, player u's optimal control in that DG is greater in absolute value and closer to the boundary of admissible values (see Fig. 1). As a result, at the final instant of time the absolute value of the phase vector in the second DG is less than in the first (see Fig. 2).

In these DGs, the terminal term of the performance index equals the squared length of the phase vector. The derivative of the performance index with respect to the phase vector will therefore decrease as the absolute value of the phase vector decreases. Since in the second DG the absolute value of the phase vector at the final instant of time is smaller, the control of player υ in the second DG has less influence on the value of the terminal term than in the first. In the second DG, therefore, it is more advantageous for player υ to choose a control of smaller absolute value than in the first game. In this way player υ can decrease the penalty term of the performance index. It can be seen in Fig. 1 that the absolute value of the optimal control of player υ in the second DG is indeed less than in the first.

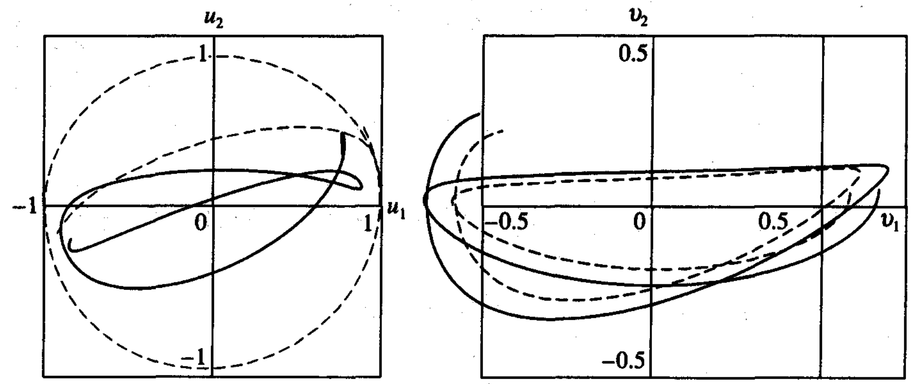

Fig. 1

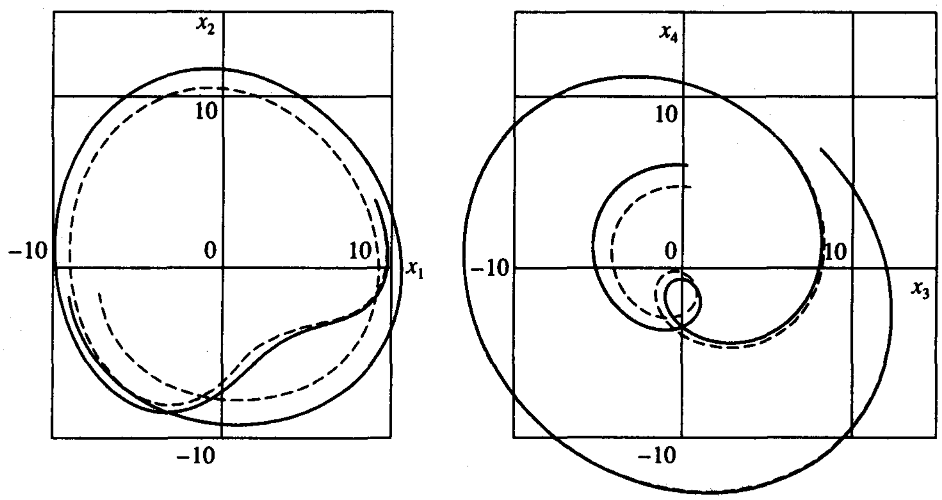

Fig. 2

I wish to dedicate this paper to the eightieth birthday of Academic N. N. Krasovskii.
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